ON THE OPTIMUM SHAPE OF APERTURES FOR A PERFORATED PLATE SUBJECT TO BENDING

V. M. Mirsalimov

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The problem of finding the optimum shape of the holes in a perforated plate weakened by a triangular or square lattice of holes and subject to bending is considered by methods based on the theory of functions of a complex variable. The criterion determining the optimum shape of the hole is based on the condition that no stress concentration should occur on the hole contour or, alternatively, that a plastic region should be created around the whole contour of the hole at exactly the same instant.

1. In order to prevent stress concentrations from arising in solid objects, it is especially interesting to discover a surface contour which will not exhibit any propensity toward brittle fracture or plastic deformation in individual regions.

Let us remind ourselves of the theory of bending as it applies to rigid (stiff) plates [1].

The displacement w of a plate normal to its surface satisfies the equation

$$\Delta\Delta w = q(x, y)/D \tag{1.1}$$

Here $D = Eh^3/12(1-\nu^2)$ is the cylindrical rigidity of the plate, q(x, y) is the transverse load, h is the thickness of the plate, E and ν are the elastic modulus and Poisson coefficient of the plate material, and Δ is the Laplace operator. In the case of q = 0 we have the basic representations [2]

$$\begin{gathered} M_x + M_y = -4D(1+\nu)\operatorname{Re}\Phi(z) \\ M_y - M_x + 2iH_{xy} = 2D(1-\nu)[\bar{z}\Phi'(z) + \Psi(z)],\ N \ \ - iN_y = -4D\Phi'(z) \end{gathered} \tag{1.2}$$

Here M_x, M_y, and H_{xy} are, respectively, the specific bending moment and torque, N_x and N_y are the specific transverse forces and $\Phi(z)$ and $\Psi(z)$ are analytical functions of the complex variable z = x + iy.

2. Let there be a doubly periodic triangular lattice composed of unknown curvilinear apertures (holes) having their centers at the points

$$\begin{gathered} P_{mn} = m\omega_1 + n\omega_2 \qquad (m, n = 0, \pm 1, \pm 2, \ldots) \\ \omega_1 = 2,\ \omega_2 = 2e^{1/3 i\pi} \end{gathered}$$

Let us denote the contour of the hole having its center at the point P_{mn} by L_{mn} and the region outside the contours L_{mn} by D_z.

On the unknown contour L_{mn} of the hole, the boundary conditions are

$$M_n = M_0,\ H_{nt} = 0,\ M_t = M_* = \text{const},\ N_t = 0,\ N_n = 0 \tag{2.1}$$

(t and n denote the directions of the tangent and normal to the contour of the solid object).

In the case of an elastic solid the quantity $M_* = \text{const}$ requires determination in the course of the solution. For an elastoplastic material Eq. (2.1) represents a condition imposed upon the development of the plastic zone, i.e., it amounts to the requirement that at the instant of generation the plastic zone should embrace the whole contour of the

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aperture at the same time, not penetrating into the interior of the solid. In this case $M_* = \text{const}$ is a specified quantity; for example, on the basis of the Tresca–St.Venant plasticity condition $M_* = M_0 \pm \sigma_s h^2/4$ (σ_s is the plasticity constant associated with tensile strain) if $M_t M_n \le 0$.

Let us transform to the parametric plane of the complex variable ζ by using the transformation $z = \omega(\zeta)$. The analytical function transforms the region D_z conformally into the region D_ζ in the ζ plane, this latter region comprising the outsides of the circles Γ_{mn} of radius λ having their centers at the points P_{mn}. On the basis of the equations [2]

$$M_x + M_y = M_n + M_t \quad (\zeta = \lambda e^{i\theta})$$
$$M_t - M_n + 2iH_{nt} = \frac{\zeta^2\omega'(\zeta)}{\lambda^2\overline{\omega'(\zeta)}}(M_y - M_x + 2iH_{xy}) \tag{2.2}$$

and the boundary conditions (2.1), in order to determine the three analytical functions $\varphi(\zeta) = \Phi[\omega(\zeta)]$, $\psi(\zeta) = \Psi[\omega(\zeta)]$ and $\omega(\zeta)$ we obtain a nonlinear boundary problem on Γ_{oo}

$$\operatorname{Re}\varphi(\zeta) = a \tag{2.3}$$

$$\zeta^2[\overline{\omega(\zeta)}\varphi'(\zeta) + \omega'(\zeta)\psi(\zeta)] = \lambda^2 b\overline{\omega'(\zeta)}$$
$$a = -\frac{M_0 + M_*}{4D(1+\nu)}, \qquad b = \frac{M_* - M_0}{2D(1-\nu)} \tag{2.4}$$

The boundary condition (2.4) may be given a different form.

It follows from the solution of the Dirichlet problem (2.3) that in the region D_ζ

$$\varphi(\zeta) = a \tag{2.5}$$

Allowing for (2.5), we may write the boundary conditions (2.4) on Γ_{oo} in the form

$$\zeta^2\omega'(\zeta)\psi(\zeta) = \lambda^2 b\overline{\omega'(\zeta)} \tag{2.6}$$

We seek the functions $\psi(\zeta)$ and $\omega(\zeta)$ in the form of series [3, 4],

$$\psi(\zeta) = \beta_0 + \sum_{k=0}^{\infty} \beta_{2k+2}\frac{\lambda^{2k+2}\gamma^{(2k)}(\zeta)}{(2k+1)!} \tag{2.7}$$

$$\omega(\zeta) = \zeta + \sum_{k=0}^{\infty} A_{2k+2}\frac{\lambda^{2k+2}\gamma^{(2k-1)}(\zeta)}{(2k+1)!} \tag{2.8}$$

where $\gamma(z)$ is an elliptic Weierstrass function,

$$\gamma(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left[\frac{1}{(z - P_{mn})^2} - \frac{1}{P_{mn}^2}\right]$$

Let us derive the relationships which the coefficients of the expressions (2.7) and (2.8) must satisfy. By equating the principal vector of the forces acting on the arc connecting two congruent points in D_ζ to zero we find that

$$a = -\frac{\pi(1-\nu)}{4\sqrt{3}(1+\nu)}\beta_2\lambda^2, \qquad \beta_0 = 0 \tag{2.9}$$

The symmetry conditions for a perforated plate with a triangular lattice of holes may be written as follows:

$$\varphi(\zeta e^{1/3 i\pi}) = \varphi(\zeta), \quad \psi(\zeta e^{1/3 i\pi}) = e^{-2/3 i\pi}\psi(\zeta)$$
$$\omega(\zeta e^{1/3 i\pi}) = e^{1/3 i\pi}\omega(\zeta)$$

and reduced to the equations

$$\beta_{6k+2\pm 2} = A_{6k\pm 2} = 0 \quad \text{for} \quad k = 0, 1, \ldots \tag{2.10}$$

In order to set up equations for the remaining coefficients on the presentations (2.7) and (2.8) of the functions $\psi(\zeta)$ and $\omega(\zeta)$, we expand these functions in Laurent series in the neighborhood of the point $\zeta = 0$

$$\psi(\zeta) = \sum_{k=0}^{\infty} \beta_{6k+2}\left(\frac{\lambda}{\zeta}\right)^{6k+2} + \sum_{k=0}^{\infty} \beta_{6k+2}\lambda^{6k+2}\sum_{j=0}^{\infty} r_{3j+2,3k}\zeta^{6j+4} \tag{2.11}$$

TABLE 1

Coeff. of the unknown functions	λ					
	0.2	0.3	0.4	0.5	0.6	0.7
			First approximation			
A_6	0.00003	0.00033	0.00188	0.00716	0.02136	0.05374
β_2/M_1	—1.03777	—1.08920	—1.17040	—1.29441	—1.48620	—1.79550
β_8/M_1	0.00003	0.00036	0.00219	0.00926	0.03170	0.09551
			Second approximation			
A_6	0.00003	0.00033	0.00188	0.00716	0.02137	0.05392
A_{12}	0.00000	0.00001	0.00065	0.00025	0.00075	0.00188
β_2/M_1	—1.03777	—1.08920	—1.17040	—1.29441	—1.48620	—1.79547
β_8/M_1	0.00003	0.00036	0.00219	0.00926	0.03171	0.09585
β_{14}/M_1	0.00000	0.00001	0.00007	0.00026	0.00043	—0.00187

$$\omega(\zeta) = \zeta - \sum_{k=1}^{\infty} \frac{A_{6k}\lambda}{6k-1}\left(\frac{\lambda}{\zeta}\right)^{6k-1} + \sum_{k=1}^{\infty} A_{6k}\lambda^{6k} \sum_{j=0}^{\infty} \frac{r_{3j,\,3k-1}}{6j+1}\zeta^{6j+1}$$
$$r_{jk} = \frac{(2j+2k+1)!\,g_{j+k+1}}{(2j)!\,(2k+1)!\,2^{2j+2k+2}} \tag{2.12}$$
$$g_{j+k+1} = \sum_{m,n}{}' \frac{1}{T^{2j+2k+2}}, \qquad T = \frac{1}{2}P_{mn}$$

In the boundary condition (2.6) for the contour Γ_{00} ($\zeta = \lambda e^{i\theta}$) we now substitute the corresponding Laurent series for $\psi(\zeta)$, $\omega'(\zeta)$ and $\overline{\omega'(\zeta)}$ and compare the coefficients of $e^{6k\theta}$ ($k = 0, \pm 1, \pm 2, \ldots$); we obtain an infinite system of nonlinear algebraic equations in β_{6k+2}, A_{6k}. The equations of the first approximation take the form

$$c\beta_2 + A_6\gamma_0 + A_6\beta_8\lambda^{12}r_{32} = bc, \; c\beta_8 + A_6\beta_2 = bA_6\lambda^{12}r_{32}$$
$$c\gamma_0 + A_6\gamma_1 + A_6\beta_2\lambda^{12}r_{32} = bA_6, \; c = 1 + A_6\lambda^6 r_{02} \tag{2.13}$$
$$\gamma_j = \beta_2 r_{3j+2,0}\lambda^{6j+6} + \beta_8 r_{3j+2,3}\lambda^{6j+12} \; (j = 0, 1)$$

The results of calculations carried out in the two first approximations are given in Table 1, in which $M_1 = M_0/D(1-\nu)$.

If in Eq. (2.12) we put $\zeta = \lambda e^{i\theta}$ we obtain the equation for the optimum shape of the hole,

$$R = |\,\omega(\lambda e^{i\theta})\,| = f(\theta) \tag{2.14}$$

In the first approximation

$$R^2 = \lambda^2(d + d_1\cos 6\theta), \quad d = c^2 + \left(\frac{1}{25} + \frac{1}{49}\lambda^{24}r_{32}^{\,2}\right)A_6^{\,2}$$
$$d_1 = 2cA_6\left(\frac{1}{7}\lambda^{12}r_{32} - \frac{1}{5}\right) \tag{2.15}$$

The constant M_* equals

$$M_* = \frac{\pi}{\sqrt{3}}D(1-\nu)\beta_2\lambda^2 - M_0 \tag{2.16}$$

For an elastoplastic plate Eq. (2.16) is the condition for the solubility of the original problem.

3. Let there be a doubly periodic square lattice with unknown curvilinear holes having their centers at the points

$$P_{mn} = m\omega_1 + n\omega_2 \; (m, n = 0, \pm 1, \pm 2, \ldots)$$
$$\omega_1 = 2, \quad \omega_2 = 2i$$

Let us consider the problem of finding the optimum shape of the hole in the square lattice. In order to obtain the solution we must repeat the discussions of Sec. 2.

We derive the solution

$$\varphi(\zeta) = -\frac{M_0 + M_*}{4D(1+\nu)} = -\frac{\pi}{8}\,\frac{1-\nu}{1+\nu}\,\beta_2\lambda^2 \tag{3.1}$$

TABLE 2

Coeffs. of the unknown functions	λ					
	0.2	0.3	0.4	0.5	0.6	0.7
			First approximation			
A_4	0.00095	0.00478	0.01513	0.03694	0.07668	0.14250
β_2/M_1	−1.03305	−1.07756	−1.14649	−1.24479	−1.39320	−1.59478
β_6/M_1	0.00097	0.00516	0.01733	0.04597	0.10558	0.21805
			Second approximation			
A_4	0.00097	0.00515	0.01733	0.04605	0.10671	0.22750
A_8	0.00000	0.00000	0.00008	0.00045	0.00197	0.00700
β_2/M_1	−1.03305	−1.07756	−1.14644	−1.24736	−1.38893	−1.56897
β_6/M_1	0.00100	0.00555	0.01986	0.05730	0.14658	0.34295
β_{10}/M_1	−0.00000	−0.00002	−0.00026	−0.00207	−0.01292	−0.06713

The functions $\psi(\zeta)$ and $\omega(\zeta)$ are defined by the series (2.7) and (2.8). Thus, we have

$$\beta_0 = 0, \ \beta_{4k} = A_{4k+2} = 0 \ \text{ for } \ k = 0, 1, \ldots \tag{3.2}$$

The results of calculations carried out in the first two approximations are given in Table 2.

The constant M_* equals

$$M_* = \frac{\pi}{2} D(1-\nu)\beta_2\lambda^2 - M_0 \tag{3.3}$$

The equation for the optimum shape of the hole in the first approximation takes the form

$$\begin{aligned} &R^2 = \lambda^2(d + d_1\cos 4\theta), \quad d = c^2 + A_4^2\left(\frac{1}{9} + \frac{1}{25}\lambda^{16} r_{21}^2\right) \\ &d_1 = 2cA_4\left(\frac{1}{5}\lambda^8 r_{21} - \frac{1}{3}\right), \quad c = 1 + A_4\lambda^4 r_{01} \end{aligned} \tag{3.4}$$

LITERATURE CITED

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